

UNIQUENESS THEOREMS FOR MEROMORPHIC MAPPINGS SHARING HYPERPLANES IN GENERAL POSITION

TING-BIN CAO AND HONG-XUN YI

ABSTRACT. The purpose of this article is to study the uniqueness problem for meromorphic mappings from \mathbb{C}^n into the complex projective space $\mathbb{P}^N(\mathbb{C})$. By making use of the method of dealing with multiple values due to L. Yang and the technique of Dethloff-Quang-Tan respectively, we obtain two general uniqueness theorems which improve and extend some known results of meromorphic mappings sharing hyperplanes in general position.

1. INTRODUCTION AND MAIN RESULTS

For a nonconstant meromorphic function f on \mathbb{C}^n and $a \in \mathbb{P}^1(\mathbb{C})$, we denote by ν_{f-a} the map from \mathbb{C} into \mathbb{Z} whose value $\nu_{f-a}(z)$ is the multiplicity of the zero of $f - a$ at z .

In 1926, R. Nevanlinna [9] proved the well-known five-value theorem that for two nonconstant meromorphic functions f and g on the complex plane \mathbb{C} , if they have the same inverse images (ignoring multiplicities) for five distinct values in $\mathbb{P}^1(\mathbb{C})$, then $f(z) \equiv g(z)$. We know that the number five of distinct values in Nevanlinna's five-value theorem cannot be reduced to four. For example, $f(z) = e^z$ and $g(z) = e^{-z}$ share four values $0, 1, -1, \infty$ (ignoring multiplicities), but $f(z) \not\equiv g(z)$. There have been several improvements of Nevanlinna's five-value theorem. H. X. Yi ([14], Theorem 3.15) adopted the method of dealing with multiple values due to L. Yang [13] and obtained a uniqueness theorem of meromorphic functions of one variable. Later, Hu, Li and Yang extended this result to meromorphic functions in several variables (see Theorem 3.9 in [8]).

Theorem 1.1. ([8], Theorem 3.9) *Let f and g be two nonconstant meromorphic functions on \mathbb{C}^n , let a_j ($j = 1, 2, \dots, q$) be q distinct complex elements in $\mathbb{P}^1(\mathbb{C})$ and take $m_j \in \mathbb{Z}^+ \cup \{\infty\}$ ($j = 1, 2, \dots, q$) satisfying $m_1 \geq m_2 \geq \dots \geq m_q$ and $\nu_{f-a_j, \leq m_j}^1 = \nu_{g-a_j, \leq m_j}^1$ ($j = 1, 2, \dots, q$). If $\sum_{j=3}^q \frac{m_j}{m_j+1} > 2$, then $f(z) \equiv g(z)$.*

Over the last few decades, there have been several generalizations of Nevanlinna's five-value theorem to the case of meromorphic mappings from \mathbb{C}^n into the complex projective space $\mathbb{P}^N(\mathbb{C})$. Some of the first results concerning this research are due to Fujimoto [6, 7].

For a meromorphic mapping f from \mathbb{C}^n into $\mathbb{P}^N(\mathbb{C})$ and a hyperplane H in $\mathbb{P}^N(\mathbb{C})$, we denote by $\nu_{(f,H)}$ the map from \mathbb{C} into \mathbb{Z} whose value $\nu_{(f,H)}(z)$ ($z \in \mathbb{C}^n$) is the intersection multiplicity of the images of f and H at $f(z)$. Let H_1, H_2, \dots ,

2000 *Mathematics Subject Classification.* 32H30, 32A22, 30D35.

Key words and phrases. meromorphic mapping, uniqueness theorem, Nevanlinna theory, hyperplane.

This work was partially supported by the NSFC (No. 10771121), the NSF of Jiangxi (No. 2008GQS0075) and the YFED of Jiangxi (No. GJJ10050) of China.

H_q be q hyperplanes in general position such that $\dim f^{-1}(H_i \cap H_j) \leq n - 2$ for $i \neq j$. Take d, m_j be positive integers or ∞ . Consider the set $\mathcal{F}_{\leq m_j}(f, \{H_j\}_{j=1}^q, d)$ of all linearly non-degenerate meromorphic mappings $g : \mathbb{C}^n \rightarrow \mathbb{P}^N(\mathbb{C})$ satisfying the conditions:

- (a) $\nu_{(f, H_j), \leq m_j}^d = \nu_{(g, H_j), \leq m_j}^d$,
- (b) $f(z) = g(z)$ on $\cup_{j=1}^q \{z \in \mathbb{C}^n \mid 0 < \nu_{(f, H_j)} \leq m_j\}$.

For brevity we will omit the notation $\leq m_j$ if $m_j = \infty$.

Fujimoto [6, 7] proved that if $q \geq 3N + 2$, then $\sharp \mathcal{F}(f, \{H_j\}_{j=1}^q, \infty) = 1$. In 1983, Smiley [11] obtained an improvement with truncated number one that if $q \geq 3N + 2$, then $\sharp \mathcal{F}(f, \{H_j\}_{j=1}^q, 1) = 1$. Later, Thai and Quang [12] considered a smaller number q and proved that if $N \geq 2$ and $q \geq 3N + 1$, then $\sharp \mathcal{F}(f, \{H_j\}_{j=1}^q, 1) = 1$. In [4], Dethloff and Tan considered the case $q \geq 2N + 3$ and obtained that if $q \geq 2N + 3$, then $\sharp \mathcal{F}(f, \{H_j\}_{j=1}^q, N) = 1$. Recently, Chen and Yan [2] improved the above results and obtained that if $q \geq 2N + 3$, then $\sharp \mathcal{F}(f, \{H_j\}_{j=1}^q, 1) = 1$.

Considering multiple values, there are some theorems. The following result is Theorem 0.1 in [1] for the special case of linearly non-degenerate meromorphic mappings.

Theorem 1.2. [1] *Let f and g be two linearly non-degenerate meromorphic mappings of \mathbb{C}^n into $\mathbb{P}^N(\mathbb{C})$, and let H_1, H_2, \dots, H_q be q hyperplanes in general position such that $\dim f^{-1}(H_i \cap H_j) \leq n - 2$ for $i \neq j$. Take $m_j (j = 1, 2, \dots, q)$ be positive integers or ∞ such that $m_1 \geq m_2 \geq \dots \geq m_q \geq N$,*

$$\nu_{(f, H_j), \leq m_j}^1 = \nu_{(g, H_j), \leq m_j}^1 \quad (j = 1, 2, \dots, q),$$

and $f(z) = g(z)$ on $\cup_{j=1}^q \{z \in \mathbb{C}^n \mid 0 < \nu_{(f, H_j)} \leq m_j\}$. If

$$(1) \quad \sum_{j=3}^q \frac{m_j}{m_j + 1} > \frac{Nq - q + N + 1}{N} + \frac{m_1}{m_1 + 1} - \frac{m_2}{m_2 + 1},$$

then $f(z) \equiv g(z)$.

Remark that condition (1) implies $q \geq 3N + 2$. In 2006, Dethloff and Tan [5] obtained the following result for a smaller q .

Theorem 1.3. [5] *If $N \geq 7$, then $\sharp \mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^{3N-1}, 1) = 1$, where $m \geq 6N + 32 + \frac{192}{N-6}$. If $4 \geq N \geq 6$, then $\sharp \mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^{3N}, 2) = 1$, where $m \geq 6N + 17 + \frac{51}{N-3}$.*

The main purpose of this paper is to consider the multiple values and uniqueness problem of meromorphic mappings. By making use of the method of dealing with multiple values due to L. Yang [13], we obtain the first main result below.

Theorem 1.4. *Let f and g be two linearly non-degenerate meromorphic mappings of \mathbb{C}^n into $\mathbb{P}^N(\mathbb{C})$, and let H_1, H_2, \dots, H_q be q hyperplanes in general position such that $\dim f^{-1}(H_i \cap H_j) \leq n - 2$ for $i \neq j$. Take $m_j (j = 1, 2, \dots, q)$ be positive integers or ∞ such that $m_1 \geq m_2 \geq \dots \geq m_q \geq N$,*

$$\nu_{(f, H_j), \leq m_j}^1 = \nu_{(g, H_j), \leq m_j}^1 \quad (j = 1, 2, \dots, q),$$

and $f(z) = g(z)$ on $\cup_{j=1}^q \{z \in \mathbb{C}^n \mid 0 < \nu_{(f, H_j)} \leq m_j\}$. If

$$(2) \quad \sum_{j=3}^q \frac{m_j}{m_j + 1} > \frac{Nq - q + N + 1}{N} - (N - 1) \left(\frac{1}{m_1 + 1} + \frac{1}{m_2 + 1} \right),$$

then $f(z) \equiv g(z)$.

Obviously, Theorem 1.4 is an improvement of Theorem 1.2. For the case $N = 1$, the condition (2) reduces to $\sum_{j=3}^q \frac{m_j}{m_j+1} > 2$. Thus Theorem 1.4 is an extension of Theorem 1.1. Furthermore, we immediately get the following corollaries from which we see that Theorem 1.4 is an improvement of Smiley's $3N + 2$ hyperplanes uniqueness theorem [11].

Corollary 1.1. *If $q \geq 3N + 2$, then $\sharp\mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^q, 1) = 1$, where $m > N - 1 + \frac{N(N+1)}{q-3N-1}$.*

Corollary 1.2. *$\sharp\mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^{3N+2}, 1) = 1$, where $m > N^2 + 2N - 1$.*

Considering a smaller q than $3N + 2$, we have another main theorem by using the technique shown in [2, 3, 4].

Theorem 1.5. *Let f and g be two linearly non-degenerate meromorphic mappings of \mathbb{C}^n into $\mathbb{P}^N(\mathbb{C})$, and let H_1, H_2, \dots, H_q be q ($q \geq 2N$) hyperplanes in general position such that $\dim f^{-1}(H_i \cap H_j) \leq n - 2$ for $i \neq j$. Take m_j ($j = 1, 2, \dots, q$) be positive integers or ∞ such that $m_1 \geq m_2 \geq \dots \geq m_q \geq N$,*

$$\nu_{(f, H_j), \leq m_j}^1 = \nu_{(g, H_j), \leq m_j}^1 \quad (j = 1, 2, \dots, q),$$

and $f(z) = g(z)$ on $\cup_{j=1}^q \{z \in \mathbb{C}^n \mid 0 < \nu_{(f, H_j)} \leq m_j\}$. If

$$(3) \quad \sum_{j=3}^q \frac{m_j}{m_j+1} > \frac{Nq - q + N + 1}{N} - \frac{4N - 4}{q + 2N - 2} + \left(\frac{1}{m_1 + 1} + \frac{1}{m_2 + 1} \right),$$

then $f(z) \equiv g(z)$.

Thus we obtain immediately the following corollaries which improve the above-mentioned uniqueness theorems for meromorphic mappings sharing hyperplanes in general position [6, 11, 12, 4, 5, 2].

Corollary 1.3. *If $q \geq 2N + 3$, then $\sharp\mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^q, 1) = 1$, where*

$$m > \frac{(N-1)q^2 + (2N^2 - N + 3)q + 2N^2 - 2}{(q + N - 1)(q - 2N - 2)}.$$

Corollary 1.4. *$\sharp\mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^{2N+3}, 1) = 1$, where $m > \frac{8N^3 + 14N^2 - 2}{3N + 2}$.*

Corollary 1.5. *If $N \geq 2$, then $\sharp\mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^{3N+1}, 1) = 1$, where $m > \frac{15N^2 - 2N + 3}{4(N-1)}$.*

Corollary 1.6. *If $N \geq 3$, then $\sharp\mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^{3N}, 1) = 1$, where $m > \frac{15N^3 - 10N^2 + 9N - 2}{(4N-1)(N-2)}$.*

Corollary 1.7. *If $N \geq 4$, then $\sharp\mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^{3N-1}, 1) = 1$, where $m > \frac{15N^3 - 18N^2 + 17N - 6}{(4N-2)(N-3)}$.*

The last corollary is a supplement of Corollary 1.2.

Corollary 1.8. *$\sharp\mathcal{F}_{\leq m}(f, \{H_j\}_{j=1}^{3N+2}, 1) = 1$, where $m > \frac{15N^2 + 6N - 1}{4N + 1}$.*

For the case $N = 1$, the condition (3) reduces to $\sum_{j=3}^q \frac{m_j}{m_j+1} > 2 + \left(\frac{1}{m_1+1} + \frac{1}{m_2+1} \right)$. Thus compared with the conditions of Theorems 1.1 and 1.4, there maybe exist a better lower estimate than condition (3) in Theorem 1.5.

2. PRELIMINARIES

We set $\|z\| = (\sum_{j=1}^n |z_j|^2)^{\frac{1}{2}}$ for $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. For $r > 0$, define $B(r) = \{z \in \mathbb{C}^n : \|z\| < r\}$, $S(r) = \{z \in \mathbb{C}^n : \|z\| = r\}$, $d^c = (4\pi\sqrt{-1})^{-1}(\partial - \bar{\partial})$, $v = (dd^c\|z\|^2)^{n-1}$ and $\sigma = d^c \log\|z\|^2 \wedge (dd^c\|z\|^2)^{n-1}$.

Let h be a nonzero entire function on \mathbb{C}^n . For $a \in \mathbb{C}^n$, we can write h as $h(z) = \sum_{m=0}^{\infty} P_m(z-a)$, where the term $P_m(z)$ is either identically zero or a homogeneous polynomial of degree m . The number $\nu_h(a) := \min\{m : P_m \neq 0\}$ is said to be the zero-multiplicity of h at a . Set $\text{supp}\nu_h := \{z \in \mathbb{C}^n : \nu_h(z) \neq 0\}$.

Let φ be a nonzero meromorphic function on \mathbb{C}^n . For each $a \in \mathbb{C}^n$, we choose nonzero holomorphic functions φ_0 and φ_1 on a neighborhood U of a such that $\varphi = \frac{\varphi_0}{\varphi_1}$ on U and $\dim(\varphi_0^{-1} \cap \varphi_1^{-1}(0)) \leq n-2$, and we define $\nu_\varphi := \nu_{\varphi_0}$, $\nu_\varphi^\infty := \nu_{\varphi_1}$, which are independent of choices of φ_0 and φ_1 .

Let f be a nonconstant meromorphic mapping of \mathbb{C}^n into $\mathbb{P}^N(\mathbb{C})$. We can choose holomorphic functions f_0, f_1, \dots, f_N on \mathbb{C}^n such that $I_f := \{z \in \mathbb{C}^n : f_0(z) = \dots = f_N(z) = 0\}$ is of dimension at most $n-2$ and $f = (f_0 : \dots : f_N)$. Usually, $(f_0 : \dots : f_N)$ is a reduced representation of f . The characteristic function of f is defined by

$$T(r, f) = \int_{S(r)} \log \|f\| \sigma - \int_{S(1)} \log \|f\| \sigma \quad (r > 1),$$

where $\|f\| = (\sum_{j=0}^N |f_j|^2)^{\frac{1}{2}}$. Note that $T(r, f)$ is independent of the choice of the reduced representation of f .

For a hyperplane $H = \{(x_0 : \dots : x_N) \in \mathbb{P}^N(\mathbb{C}) : a_0 x_0 + \dots + a_N x_N = 0\}$, we say that (f, H) is free if $(f, H) = \sum_{j=0}^N a_j f_j \neq 0$. Under the assumption that (f, H) is free, (f, H) is a nonzero holomorphic function and the proximity function of f and H is defined by

$$m_{f,H}(r) = \int_{S(r)} \log \frac{\|f\| \|H\|}{|(f, H)|} \sigma - \int_{S(1)} \log \frac{\|f\| \|H\|}{|(f, H)|} \sigma, \quad r > 1,$$

where $\|H\| = (\sum_{j=0}^N |a_j|^2)^{\frac{1}{2}}$. The proximity function of a meromorphic function φ on \mathbb{C}^n is defined by $m(r, \varphi) = \int_{S(r)} \log^+ |\varphi| \sigma$, where $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$.

Let k, M be positive integers or $+\infty$. For a divisor ν on \mathbb{C}^n . We define the counting functions of ν as follows. Set

$$\nu^M(z) = \min\{\nu(z), M\}, \quad \nu_{\leq k}^M(z) = \begin{cases} 0, & \text{if } \nu(z) > k; \\ \nu^M(z), & \text{if } \nu(z) \leq k, \end{cases}$$

$$\nu_{\geq k}^M(z) = \begin{cases} 0, & \text{if } \nu(z) < k; \\ \nu^M(z), & \text{if } \nu(z) \geq k. \end{cases}$$

and

$$n(t) = \begin{cases} \int_{\text{supp}\nu \cap B(t)} \nu(z) v, & \text{if } n \geq 2; \\ \sum_{|z| \leq t} \nu(z), & \text{if } n = 1. \end{cases}$$

Similarly, we define $n^M(t)$, $n_{\geq k}^M(t)$ and $n_{\leq k}^M(t)$. We define

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2n-1}} dt \quad (r > 1).$$

Similarly, we define $N(r, \nu^M)$, $N(r, \nu_{\leq k}^M)$ and $N(r, \nu_{\geq k}^M)$ and denote them by $N^M(r, \nu)$, $N_{\leq k}^M(r, \nu)$ and $N_{\geq k}^M(r, \nu)$, respectively.

For a meromorphic function φ on \mathbb{C}^n , we denote by

$$\begin{aligned} N_\varphi(r) &= N(r, \nu_\varphi), & N_\varphi^M(r) &= N^M(r, \nu_\varphi), \\ N_{\varphi, \leq k}^M(r) &= N_{\leq k}^M(r, \nu_\varphi), & N_{\varphi, \geq k}^M(r) &= N_{\geq k}^M(r, \nu_\varphi). \end{aligned}$$

For brevity we will omit the superscript M if $M = \infty$.

For a closed subset A of a purely $(n-1)$ -dimensional analytic subset of \mathbb{C}^n , we define

$$n_A^1(t) = \begin{cases} \int_{A \cap B(t)} v, & \text{if } n \geq 2; \\ \#(A \cap B(t)), & \text{if } n = 1, \end{cases}$$

and

$$N^1(r, A) = \int_1^r \frac{n_A^1(t)}{t^{2n-1}} dt \quad (r > 1).$$

The First Main Theorem is that

$$T(r, f) = N_{(f, H)}(r) + m_{f, H}(r) + O(1).$$

As usual, by the notation " $\|P$ " we mean the assertion P holds for all $r > 1$ excluding a set of finite Lebesgue measure.

Theorem 2.1. (*Second Main Theorem*) *Let f be a linearly non-degenerate meromorphic mapping from \mathbb{C}^n into $\mathbb{P}^N(\mathbb{C})$ and H_1, \dots, H_q be q hyperplanes in general position. Then*

$$\|(q - N - 1)T(r, f) \leq \sum_{j=1}^q N_{(f, H_j)}^N(r) + o(T(r, f)).$$

For two distinct hyperplanes H_1 and H_2 , we have

Lemma 2.1. [10] $T(r, \frac{(f, H_1)}{(f, H_2)}) \leq T(r, f) + O(1)$.

The following lemma is proved by using the method of dealing with multiple values due to L. Yang [13], see also Lemma 4.7 in [12].

Lemma 2.2. *Let f be a linearly non-degenerate meromorphic mapping from \mathbb{C}^n into $\mathbb{P}^N(\mathbb{C})$, H be a hyperplane in general position, and $k(\geq N)$ be a positive integer. Then*

$$N_{(f, H)}^N(r) \leq N(1 - \frac{N}{k+1})N_{(f, H), \leq k}^1(r) + \frac{N}{k+1}N_{(f, H)}(r);$$

and

$$N_{(f, H)}^N(r) \leq N(1 - \frac{N}{k+1})N_{(f, H), \leq k}^1(r) + \frac{N}{k+1}T(r, f) + o(T(r, f)).$$

Proof. From

$$N_{(f, H)}^N(r) = N_{(f, H), \leq k}^N(r) + N_{(f, H), \geq k+1}^N(r)$$

and

$$N_{(f, H), \geq k+1}^N(r) \leq \frac{N}{k+1}N_{(f, H), \geq k+1}(r) \leq \frac{N}{k+1} \left(N_{(f, H)}(r) - N_{(f, H), \leq k}^N(r) \right),$$

we deduce that

$$\begin{aligned} N_{(f,H)}^N(r) &\leq \left(1 - \frac{N}{k+1}\right) N_{(f,H),\leq k}^N(r) + \frac{N}{k+1} N_{(f,H)}(r) \\ &\leq N \left(1 - \frac{N}{k+1}\right) N_{(f,H),\leq k}^1(r) + \frac{N}{k+1} N_{(f,H)}(r). \end{aligned}$$

This completes the proof of the first inequality of the lemma. The second inequality of the lemma follows immediately because of $N_{(f,H)}(r) \leq T(r, f) + o(T(r, f))$. \square

3. PROOF OF THEOREM 1.4

Suppose that $f(z) \not\equiv g(z)$. By the Second Main Theorem, we have

$$\|(q - N - 1)T(r) \leq \sum_{j=1}^q \left(N_{(f,H_j)}^N(r) + N_{(g,H_j)}^N(r) \right) + o(T(r)),$$

where $T(r) = T(r, f) + T(r, g)$. By Lemma 2.2 we have

$$\begin{aligned} N_{(f,H_j)}^N(r) + N_{(g,H_j)}^N(r) &\leq N \left(1 - \frac{N}{m_j+1}\right) \left(N_{(f,H_j),\leq m_j}^1(r) + N_{(g,H_j),\leq m_j}^1(r) \right) \\ &\quad + \frac{N}{m_j+1} T(r) + o(T(r)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|(q - N - 1)T(r) &\leq N \sum_{j=1}^q \left(1 - \frac{N}{m_j+1}\right) \left(N_{(f,H_j),\leq m_j}^1(r) + N_{(g,H_j),\leq m_j}^1(r) \right) \\ &\quad + N \sum_{j=1}^q \frac{1}{m_j+1} T(r) + o(T(r)). \end{aligned}$$

Noting that $m_1 \geq m_2 \geq \dots \geq m_q \geq N$, we have

$$\begin{aligned} &\sum_{j=1}^q \left(1 - \frac{N}{m_j+1}\right) N_{(f,H_j),\leq m_j}^1(r) \\ &= \left(1 - \frac{N}{m_1+1}\right) N_{(f,H_1),\leq m_1}^1(r) + \sum_{j=2}^q \left(1 - \frac{N}{m_j+1}\right) N_{(f,H_j),\leq m_j}^1(r) \\ &\leq \left(1 - \frac{N}{m_1+1}\right) N_{(f,H_1),\leq m_1}^1(r) + \sum_{j=2}^q \left(1 - \frac{N}{m_2+1}\right) N_{(f,H_j),\leq m_j}^1(r) \\ &= \left(\frac{N}{m_2+1} - \frac{N}{m_1+1}\right) N_{(f,H_1),\leq m_1}^1(r) + \sum_{j=1}^q \left(1 - \frac{N}{m_2+1}\right) N_{(f,H_j),\leq m_j}^1(r) \\ &\leq \left(\frac{N}{m_2+1} - \frac{N}{m_1+1}\right) T(r, f) + \sum_{j=1}^q \left(1 - \frac{N}{m_2+1}\right) N_{(f,H_j),\leq m_j}^1(r) + O(1). \end{aligned}$$

The same is true for g instead of f . Hence, we can deduce that

$$\begin{aligned} & \|(q - N - 1)T(r) \\ & \leq N^2 \left(\frac{1}{m_2 + 1} - \frac{1}{m_1 + 1} \right) T(r) + N \sum_{j=1}^q \frac{1}{m_j + 1} T(r) \\ & \quad + N \left(1 - \frac{N}{m_2 + 1} \right) \sum_{j=1}^q \left(N_{(f, H_j), \leq m_j}^1(r) + N_{(g, H_j), \leq m_j}^1(r) \right) + o(T(r)). \end{aligned}$$

Since $\nu_{(f, H_j), \leq m_j}^1 = \nu_{(g, H_j), \leq m_j}^1$ and $f(z) = g(z)$ on $\cup_{j=1}^q \{z \in \mathbb{C}^n | 0 < \nu_{(f, H_j)} \leq m_j\}$, we have

$$\sum_{j=1}^q \left(N_{(f, H_j), \leq m_j}^1(r) + N_{(g, H_j), \leq m_j}^1(r) \right) \leq 2N_{f-g}(r) \leq 2T(r) + O(1).$$

Hence, we can deduce that

$$\begin{aligned} \|(q - N - 1)T(r) & \leq \left(N \sum_{j=1}^q \frac{1}{m_j + 1} + 2N - N^2 \left(\frac{1}{m_1 + 1} + \frac{1}{m_2 + 1} \right) \right) T(r) \\ & \quad + o(T(r)). \end{aligned}$$

Noting that $q = \sum_{j=1}^q \frac{m_j + 1}{m_j + 1}$, we can obtain from the above inequality that

$$\left\| \left(\sum_{j=3}^q \frac{m_j}{m_j + 1} - \frac{Nq - q + N + 1}{N} + (N - 1) \left(\frac{1}{m_1 + 1} + \frac{1}{m_2 + 1} \right) \right) T(r) \right\| \leq o(T(r)).$$

This is a contradiction.

4. PROOF OF THEOREM 1.5

For brevity we denote $T(r, f) + T(r, g)$ by $T(r)$. Suppose that $f(z) \not\equiv g(z)$. Then by changing indices, if necessary, we may assume that

$$\begin{aligned} & \underbrace{\frac{(f, H_1)}{(g, H_1)} \equiv \frac{(f, H_2)}{(g, H_2)} \equiv \dots \equiv \frac{(f, H_{k_1})}{(g, H_{k_1})}}_{\text{group 1}} \not\equiv \underbrace{\frac{(f, H_{k_1+1})}{(g, H_{k_1+1})} \equiv \dots \equiv \frac{(f, H_{k_2})}{(g, H_{k_2})}}_{\text{group 2}} \\ & \not\equiv \dots \not\equiv \underbrace{\frac{(f, H_{k_{s-1}+1})}{(g, H_{k_{s-1}+1})} \equiv \dots \equiv \frac{(f, H_{k_s})}{(g, H_{k_s})}}_{\text{group s}}, \end{aligned}$$

where $k_s = q$. Then the number of elements of every group is at most N because $f(z) \not\equiv g(z)$.

The map $\sigma : \{1, 2, \dots, q\} \rightarrow \{1, 2, \dots, q\}$ is defined by

$$\sigma(i) = \begin{cases} i + N, & \text{if } i + N \leq q; \\ i + N - q, & \text{if } i + N > q. \end{cases}$$

Obviously, σ is bijective. Since $q \geq 2N$, $|\sigma(i) - i| \geq N$. Thus, $\frac{(f, H_i)}{(g, H_i)}$ and $\frac{(f, H_{\sigma(i)})}{(g, H_{\sigma(i)})}$ belongs to distinct group, and so $\frac{(f, H_i)}{(g, H_i)} \not\equiv \frac{(f, H_{\sigma(i)})}{(g, H_{\sigma(i)})}$.

We here use the technique in [3], see also [2, 4]. Set $P_i := (f, H_i)(g, H_{\sigma(i)}) - (f, H_{\sigma(i)})(g, H_i) \not\equiv 0$, where $1 \leq i \leq q$. By the assumption and the definition of

function P_i we get that for $k \in \{i, \sigma(i)\}$ and any $z_0 \in \{z \in |1 \leq \nu_{(f, H_k)} \leq m_k\}$ ($= \{z \in |1 \leq \nu_{(g, H_k)} \leq m_k\}$), z_0 is a zero of P_i with

$$\nu_{P_i}(z_0) \geq \min\{\nu_{(f, H_k)}(z_0), \nu_{(g, H_k)}(z_0)\}$$

outside an analytic set of codimension ≥ 2 . On the other hand, since $\nu_{(f, H_k), \leq m_k}^1 = \nu_{(g, H_k), \leq m_k}^1$ we have

$$\min\{\nu_{(f, H_k)}(z_0), \nu_{(g, H_k)}(z_0)\} \geq \nu_{(f, H_k), \leq m_k}^N(z_0) + \nu_{(g, H_k), \leq m_k}^N(z_0) - N\nu_{(f, H_k), \leq m_k}^1(z_0).$$

We also get that for any $j \in \{1, 2, \dots, q\} \setminus \{i, \sigma(i)\}$, any zero of (f, H_j) is also a zero of P_i outside an analytic set of codimension ≥ 2 . Thus we have

$$\begin{aligned} \nu_{P_i} &\geq \nu_{(f, H_i), \leq m_i}^N + \nu_{(f, H_{\sigma(i)}), \leq m_{\sigma(i)}}^N + \nu_{(g, H_i), \leq m_i}^N + \nu_{(g, H_{\sigma(i)}), \leq m_{\sigma(i)}}^N \\ &\quad - N\nu_{(f, H_i), \leq m_i}^1 - N\nu_{(f, H_{\sigma(i)}), \leq m_{\sigma(i)}}^1 + \sum_{j=1, j \neq i, \sigma(i)}^q \nu_{(f, H_j), \leq m_j}^1 \end{aligned}$$

outside an analytic set of codimension ≥ 2 . Hence, for all $i \in \{1, 2, \dots, q\}$ we have

$$\begin{aligned} N_{P_i} &\geq N_{(f, H_i), \leq m_i}^N(r) + N_{(f, H_{\sigma(i)}), \leq m_{\sigma(i)}}^N(r) + N_{(g, H_i), \leq m_i}^N(r) + N_{(g, H_{\sigma(i)}), \leq m_{\sigma(i)}}^N(r) \\ &\quad - NN_{(f, H_i), \leq m_i}^1(r) - NN_{(f, H_{\sigma(i)}), \leq m_{\sigma(i)}}^1(r) + \sum_{j=1, j \neq i, \sigma(i)}^q N_{(f, H_j), \leq m_j}^1(r). \end{aligned}$$

On the other hand, by Jensen's formula we have

$$\begin{aligned} N_{P_i}(r) &= \int_{S(r)} \log |P_i| \sigma + O(1) \\ &\leq \int_{S(r)} \log(|(f, H_i)|^2 + |(f, H_{\sigma(i)})|^2)^{\frac{1}{2}} \sigma \\ &\quad + \int_{S(r)} \log(|(g, H_i)|^2 + |(g, H_{\sigma(i)})|^2)^{\frac{1}{2}} \sigma + O(1) \\ &\leq T(r) + O(1). \end{aligned}$$

Therefore, for all $i \in \{1, 2, \dots, q\}$ we have

$$\begin{aligned} &T(r) + O(1) \\ &\geq N_{(f, H_i), \leq m_i}^N(r) + N_{(f, H_{\sigma(i)}), \leq m_{\sigma(i)}}^N(r) + N_{(g, H_i), \leq m_i}^N(r) + N_{(g, H_{\sigma(i)}), \leq m_{\sigma(i)}}^N(r) \\ &\quad - NN_{(f, H_i), \leq m_i}^1(r) - NN_{(f, H_{\sigma(i)}), \leq m_{\sigma(i)}}^1(r) + \sum_{j=1, j \neq i, \sigma(i)}^q N_{(f, H_j), \leq m_j}^1(r). \end{aligned}$$

Note that σ is bijective. Take summation of the above inequality over $1 \leq i \leq q$, we have

$$\begin{aligned} &(q - 2N - 2) \sum_{j=1}^q N_{(f, H_j), \leq m_j}^1(r) + 2 \sum_{j=1}^q \left(N_{(f, H_j), \leq m_j}^N(r) + N_{(g, H_j), \leq m_j}^N(r) \right) \\ &\leq qT(r) + O(1). \end{aligned}$$

By a similar discussion for g instead of f , we have

$$\begin{aligned} & (q - 2N - 2) \sum_{j=1}^q N_{(g, H_j), \leq m_j}^1(r) + 2 \sum_{j=1}^q \left(N_{(f, H_j), \leq m_j}^N(r) + N_{(g, H_j), \leq m_j}^N(r) \right) \\ & \leq qT(r) + O(1). \end{aligned}$$

Noting that $\frac{1}{N} N_{(f, H_j), \leq m_j}^N(r) \leq N_{(f, H_j), \leq m_j}^1(r)$, we get from the above inequalities that

$$\frac{q + 2N - 2}{2N} \sum_{j=1}^q \left(N_{(f, H_j), \leq m_j}^N(r) + N_{(g, H_j), \leq m_j}^N(r) \right) \leq qT(r) + O(1).$$

By the Second Main Theorem, we get

$$\begin{aligned} (q - N - 1)T(r) & \leq \sum_{i=1}^q \left(N_{(f, H_i), \leq m_i}^N(r) + N_{(g, H_i), \leq m_i}^N(r) \right) \\ & \quad + \sum_{i=1}^q \left(N_{(f, H_i), \geq m_i+1}^N(r) + N_{(g, H_i), \geq m_i+1}^N(r) \right) + o(T(r)). \end{aligned}$$

Therefore, removing the term $\sum_{i=1}^q \left(N_{(f, H_i), \leq m_i}^N(r) + N_{(g, H_i), \leq m_i}^N(r) \right)$ from the above inequalities we have

$$\begin{aligned} & \left(\frac{(q + 2N - 2)(q - N - 1)}{2N} - q \right) T(r) \\ & \leq \frac{q + 2N - 2}{2N} \sum_{j=1}^q \left(N_{(f, H_j), \geq m_j+1}^N(r) + N_{(g, H_j), \geq m_j+1}^N(r) \right) + o(T(r)). \end{aligned}$$

Noting that

$$\begin{aligned} & N_{(f, H_j), \geq m_j+1}^N(r) + N_{(g, H_j), \geq m_j+1}^N(r) \\ & \leq \frac{N}{m_j + 1} \left(N_{(f, H_j), \geq m_j+1}(r) + N_{(g, H_j), \geq m_j+1}(r) \right) \\ & \leq \frac{N}{m_j + 1} \left(N_{(f, H_j)}(r) + N_{(g, H_j)}(r) \right) \leq \frac{N}{m_j + 1} T(r) + o(T(r)), \end{aligned}$$

we get from the above inequality that

$$\left(\frac{(q + 2N - 2)(q - N - 1)}{2N} - q \right) T(r) \leq \frac{q + 2N - 2}{2} \sum_{j=1}^q \frac{1}{m_j + 1} T(r) + o(T(r)).$$

Noting that $q = \sum_{j=1}^q \frac{m_j+1}{m_j+1}$, we can obtain from the above inequality that

$$\begin{aligned} & \left\| \left(\sum_{j=3}^q \frac{m_j}{m_j + 1} - \frac{Nq - q + N + 1}{N} + \frac{4N - 4}{q + 2N - 2} - \left(\frac{1}{m_1 + 1} + \frac{1}{m_2 + 1} \right) \right) T(r) \right\| \\ & \leq o(T(r)). \end{aligned}$$

This is a contradiction.

Acknowledgements. The authors would like to thank Dr. Qi-Ming Yan, Dr. Hong-Zhe Cao and Dr. Kai Liu for making some valuable suggestions to improve the present paper.

REFERENCES

- [1] Y. Aihara, Unicity theorems for meromorphic mappings with deficiencies, *Complex Variables* 42(2000), 259-268.
- [2] Z. H. Chen and Q. M. Yan, Uniqueness theorem of meromorphic mappings into $\mathbb{P}^N(\mathbb{C})$ sharing $2N + 3$ hyperplanes regardless of multiplicities, *Intern. J. Math.* 20(2009), No. 6, 717-726.
- [3] G. Dethloff, S. D. Quang and T. V. Tan, A uniqueness theorem for meromorphic mappings with two families of hyperplanes, Preprint.
- [4] G. Dethloff and T. V. Tan, Uniqueness theorems for meromorphic mappings with few hyperplanes, *Bull. Sci. Math.* 133(2009), 501-514.
- [5] G. Dethloff and T. V. Tan, An extension of uniqueness theorems for meromorphic mappings, *Vietnam J. Math.* 34(2006), No. 1, 71-94.
- [6] H. Fujimoto, The uniqueness problem of meromorphic maps into the complex projective spaces, *Nagoya Math. J.* 58(1975), 1-23.
- [7] H. Fujimoto, The uniqueness theorem for algebraically non-degenerate meromorphic maps into $\mathbb{P}^N(\mathbb{C})$, *Nagoya Math. J.* 64(1976), 117-147.
- [8] P. C. Hu, P. Li and C. C. Yang, *Unicity of meromorphic mappings*, Kluwer 2003.
- [9] R. Nevanlinna, Eindentig keitssätze in der theorie der meromorphen funktionen, *Acta. Math.* 48(1926), 367-391.
- [10] M. Ru, *Nevanlinna theory and its relation to Diophantine Approximation*, Singapore: World Scientific Publishing, 2001.
- [11] L. Smiley, Geometric conditions for unicity of holomorphic curves, *Contemp. Math.* 25((1983), 149-154.
- [12] D. D. Thai and S. D. Quang, Uniqueness problem with truncated multiplicities of meromorphic mappings in several complex variables, *Intern. J. Math.* 17(2006), No. 10, 1223-1257.
- [13] L. Yang, Multiple values of meromorphic functions and functions combination, *Acta Math. Sin. Chinese Ser.* 14(1964), 428-437.(in Chinese)
- [14] H.-X. Yi and C.-C. Yang, *Uniqueness theory of meromorphic functions*, Science Press 1995/Kluwer 2003.

(Ting-Bin Cao) DEPARTMENT OF MATHEMATICS, NANCHANG UNIVERSITY, NANCHANG, JIANGXI 330031, CHINA

E-mail address: `tbcao@ncu.edu.cn`, `ctb97@163.com` (the corresponding author)

(Hong-Xun Yi) DEPARTMENT OF MATHEMATICS, SHANDONG UNIVERSITY, JINAN, SHANDONG 250100, CHINA

E-mail address: `hxyi@sdu.edu.cn`